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The Extension and Convergence of Positive Operators

R. L. JAMES

Department of Mathematics, Stanislaus State College, 800 Monte Vista Avenue, Turlock, California 95380

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1. INTRODUCTION

The ordinary Riemann integral can be regarded as an extension of the integral of a polynomial to a larger space in the following way. Let $\mathscr{B}$ denote the vector space of all bounded real-valued functions on [0, 1] and $\mathscr{P}$ the subspace consisting of the polynomials. For x in $\mathscr{P}$ define

$$P_0x = \int_0^1 x(t)\,dt.$$

Then P_0 is a linear functional defined on $\mathscr{P}$. Moreover P_0 is positive, i.e., $P_0x \geqslant 0$ for all $x \in \mathscr{P}$, $x \geqslant 0$.

The usual definition of the Riemann integral in terms of upper and lower sums yields the following.

THEOREM 1.1. *An element x of $\mathscr{B}$ is Riemann integrable if and only if for each $n = 1, 2,\ldots,$ there exist x_n, $x^n \in \mathscr{P}$ such that*

(i) $x_n(t) \leqslant x(t) \leqslant x^n(t)$, $0 \leqslant t \leqslant 1$,

(ii) $P_0(x^n - x_n) \to 0$ *as* $n \to \infty$.

If (i) and (ii) are satisfied, then the sequences $\{P_0x^n\}$, $\{P_0x_n\}$ both converge to

$$\int_0^1 x(t)\,dt.$$

If we set

$$Px = \lim_{n\to\infty} P_0x^n = \lim_{n\to\infty} P_0x_n$$

then Theorem 1.1 yields a vector space $\mathscr{R}$ (the Riemann integrable functions) such that $\mathscr{P} \subset \mathscr{R} \subset \mathscr{B}$, and P is an extension of P_0 to $\mathscr{R}$.

Apparently Theorem 1.1 was first established by H. Weyl [13]. There are some advantages in this approach to the Riemann integral. Many of the properties of polynomials or continuous functions are inherited by the Riemann integrable functions through Theorem 1.1. Numerical quadrature is a specific example of this. For each x in $\mathscr{R}$ let

$$P_n x = \sum_{k=1}^{n} w_{nk} x(t_{nk}), \qquad w_{nk} \geqslant 0, \qquad 0 \leqslant t_{nk} \leqslant 1, \qquad n = 1, 2, \ldots.$$

If $P_n x \to Px$ for each x in $\mathscr{P}$, then $P_n x \to Px$ for each x in $\mathscr{R}$ [12].

This order-approximation approach to the Riemann integral has been used by others [2, 5, 7, 8] in more recent years. In [1], Anselone replaced the Riemann integral by a positive linear functional defined on an arbitrary ordered vector space. We shall consider the more general case of a positive operator mapping an ordered vector space into an ordered topological vector space. In Section 2 we shall establish a result corresponding to Theorem 1.1 in this more general situation. This result is applied, in Section 3, to the Bochner integral of functions mapping a locally compact Hausdorff space into a Banach lattice.

In Sections 4 and 5 we consider the pointwise convergence of a sequence (or net) of positive operators to a positive operator P. We establish some results similar to those mentioned above concerning the convergence of numerical quadrature to the Riemann integral. More precisely, we prove that convergence on a suitable subspace will imply convergence on the whole space. In Section 4 a generalization of Korovkin's monotone operator theorem on $C[0, 1]$ is given and Korovkin's theorem on R^m is given. Weak convergence is considered in Section 5 and in addition a Korovkin type result is given for $L_p[0, 1]$, $1 \leqslant p < \infty$.

In this paper, all spaces will be real. By an ordered topological vector space (TVS) we mean a TVS which has an order structure. If we want the positive cone to be normal or closed we shall specifically say so. For the definition of a normal cone and related material we shall follow [9].

2. Extensions of Positive Operators

We first give a procedure that extends a positive operator defined on an ordered vector space.

Let Y be a sequentially complete, Hausdorff TVS ordered by a normal, closed positive cone.

THEOREM 2.1. *Let X_1 be an ordered vector space, X_0 a subset, and P_0 a* *positive operator mapping* X_0 *into* Y. *We define a set* X *as follows. Given* x *in* X_1, x *is in* X *if and only if, for* $n = 1, 2, \ldots$, *there exist* x_n, x^n *in* X_0 *with*

(i) $x_n \leqslant x \leqslant x^n$,

(ii) $P_0 x^n - P_0 x_n \to 0$ *as* $n \to \infty$.

Then $X_0 \subset X \subset X_1$. *The sequences* $\{P_0 x^n\}$, $\{P_0 x_n\}$ *converge to the same limit. For* x *in* X *define*

(iii) $Px = \lim_{n\to\infty} P_0 x^n = \lim_{n\to\infty} P_0 x_n$.

Then Px *is independent of the choices of* $\{x^n\}$ *and* $\{x_n\}$ *and* P *is a positive operator mapping* X *into* Y. *Moreover if* P_0 *is a linear operator and* X_0 *a subspace, then* X *is a subspace and* P *a linear operator.*

Proof. Let $\{x^n\}$, $\{x_n\}$ be sequences in X_0 such that (i) and (ii) hold. Then for any integers n and p

$$-(P_0 x^n - P_0 x_n) \leqslant P_0 x^{n+p} - P_0 x^n \leqslant P_0 x^{n+p} - P_0 x_{n+p},$$

so $\{P_0 x^n\}$ is a Cauchy sequence in Y and therefore converges. By (ii), $\{P_0 x_n\}$ converges to the same limit. If $\{y^n\}$, $\{y_n\}$ are two other sequences satisfying (i) and (ii), then

$$\lim_{n\to\infty} P_0 x^n = \lim_{n\to\infty} P_0 x_n \leqslant \lim_{n\to\infty} P_0 y^n$$

and by symmetry we have equality. The remainder of the proof is routine.

If X_1 is a TVS and X_0 is closed, then X may not be closed. To see this let $X_1 = L_1[0, 1]$, X_0 the subspace consisting of the continuous functions and P_0 the Riemann integral.

PROPOSITION 2.1. *Assume that the hypotheses of Theorem* 2.1 *hold and that* X_0, X_1 *are vector lattices. Then* X *is a vector lattice.*

Proof. This follows from the inequality

$$|(x \vee z) - (y \vee z)| \leqslant |x - y|.$$

In the next section we use Theorem 2.1 to define the Riemann–Bochner integral.

3. THE RIEMANN–BOCHNER INTEGRAL

Let $(X, \mathcal{A}, \mu)$ be a measure space and E a Banach lattice. E' will denote the normed dual of E. A function x mapping X into E is said to be *weakly measurable* if for each f in E', $f(x(t))$ is a measurable function; x is said to be *strongly measurable* if there is a sequence of finitely valued functions converging strongly to x a.e.

We will need the following two results, the first of which is due to Pettis [10] and the second to Bochner [3].

PROPOSITION 3.1. *A function x mapping X into E is strongly measurable if and only if*

(i) *x is weakly measurable and*

(ii) *there exists a set B_0 of measure zero such that $\{x(t) : t \in X - B_0\}$ is separable.*

PROPOSITION 3.2. *A strongly measurable function x is Bochner integrable if and only if $\| x(t)\|$ is integrable.*

We define a linear map P from the vector space V of strongly measurable and (Bochner) integrable functions into E by

$$Px = \int_X x \, d\mu.$$

For x in V, we write $x \geqslant 0$ if $x(t) \geqslant 0$ a.e. Suppose $x \in V$, $x \geqslant 0$. For any positive functional f in E' we have

$$f(Px) \geqslant 0$$

whence P is a positive map V into E.

LEMMA 3.1. *If $x \in V$, $x \geqslant 0$ and $Px = 0$, then $x(t) = 0$ a.e.*

Proof. By Proposition 3.1 we may assume without loss of generality that E is separable. Then there is a sequence $\{f_n\} \subset E'$, $\|f_n\| \leqslant 1$, $n = 1, 2,...$, such that for any f_0 in E', $\|f_0\| \leqslant 1$, there is a subsequence $\{f_{n'}\}$ of $\{f_n\}$ such that $f_{n'}(x) \to f_0(x)$ for all x in E [15]. Let $\epsilon > 0$ be fixed and define $A_\epsilon = \{t \in X : \| x(t)\| > \epsilon\}$, $A_f = \{t \in X : f(x(t)) > \epsilon\}$, $f \in E'$. We now show $A_\epsilon = \bigcup_{n=1}^{\infty} A_{f_n}$. It is clear that $A_\epsilon \supset \bigcup_{n=1}^{\infty} A_{f_n}$. If $t \in A_\epsilon$ then by the Hahn–Banach theorem there exists $f_0 \in E'$, $\|f_0\| = 1$, $f_0(x(t)) = \| x(t)\| > \epsilon$. There exists a subsequence $\{f_{n'}\}$ such that $f_{n'}(x(t)) \to f_0(x(t))$, whence $t \in A_{f_n}$ for some n, i.e., $A_\epsilon \subset \bigcup_{n=1}^{\infty} A_{f_n}$. For any positive functional f in E', $f(Px) = P(f(x)) = 0$ so that A_f has measure zero for any f in E'. Therefore $\mu(A_\epsilon) = 0$ for any positive ϵ, which implies $x(t) = 0$ a.e.

At this point we take X to be a locally compact topological space and μ a regular Borel measure on X. We will require E to have the property that every positive decreasing sequence is strongly convergent.

Let $C_c(X : E)$ denote the continuous functions mapping X into E with compact support, and $B(X : E)$ the bounded functions mapping X into E. For x in $B(X : E)$, define

$$\| x \| = \sup_{t \in X} \| x(t) \| .$$

With this norm $B(X : E)$ is a Banach space and $C_c(X : E)$ is a closed subspace.

Each x in $C_c(X : E)$ has separable range in E; hence x is strongly measurable and integrable. For x in $C_c(X : E)$, define

$$P_0 x = \int_X x \, d\mu.$$

Clearly P_0 is a positive, continuous map from $C_c(X : E)$ into E. By Theorem 2.1 we extend P_0 to a positive operator P defined on the subspace $R(X : E)$ and $C_c(X : E) \subset R(X : E) \subset B(X : E)$. From Proposition 2.1 we conclude that $R(X : E)$ is a vector lattice.

THEOREM 3.1. *If $x \in R(X : E)$ then x is continuous a.e. $[\mu]$.*

Proof. For each integer $m \geqslant 1$, there exist x^m, x_m in $C_c(X : E)$ such that $x_m(t) \leqslant x(t) \leqslant x^m(t)$, $x_m(t) \uparrow$, $x^m(t) \downarrow$ and $P_0(x^m - x_m) \to 0$ as $m \to \infty$. Hence there exist functions $\sup x$, $\inf x : X \to E$ such that $\sup x(t) = \lim_{m\to\infty} x^m(t)$, $\inf x(t) = \lim_{m\to\infty} x_m(t)$, $t \in X$. Since x^m, x_m, $m \geqslant 1$, have separable ranges, $\sup x$, $\inf x$ also have separable ranges and thus $\sup x$, $\inf x$ are Bochner integrable. Note that

$$0 \leqslant \int (\sup x - \inf x) \, d\mu = \int (\sup x - x^m) \, d\mu + \int (x^m - x_m) \, d\mu + \int (x_m - \inf x) \, d\mu \to 0 \text{ as } m \to \infty.$$

By Lemma 3.1, $\sup x(t) = \inf x(t)$ a.e. From the inequality

$$\| x(t_0) - x(t) \| \leqslant \| x^m(t_0) - x_m(t_0) \| + \| x_m(t_0) - x_m(t) \|$$

we see that x is continuous wherever $\sup x(t_0) = \inf x(t_0)$. Thus x is continuous almost everywhere.

It is known [2] that the converse of Theorem 3.1 is true when E is the real line (and thus for $E = R^m$) but we don't know whether it is true in general.

PROPOSITION 3.1. *If x is continuous a.e. and has compact support, then x is strongly measurable.*

Proof. Let B be the set of points at which x is not continuous. There exist open sets G_m, $m = 1, 2,\ldots$, such that $\mu(G_m) < 1/m$, $G_1 \supset G_2 \supset \cdots \supset B$. Let F be the support of x. Each of the sets $\{x(t) : t \in F - G_m\}$ is compact and hence separable, whence $\{x(t) : t \in F - \bigcap_{m=1}^{\infty} G_m\}$ is separable. Since $\bigcap_{m=1}^{\infty} G_m$ has measure zero, x is strongly measurable.

Since each x in $R(X : E)$ has compact support, x is strongly measurable and integrable.

If $X = [0, 1]$ and E is the real line, then $R(X : E)$ is the set of Riemann integrable functions on $[0, 1]$. In view of this it seems reasonable to call $R(X : E)$ the set of Riemann–Bochner integrable functions and P the *Riemann–Bochner* integral.

4. Pointwise Convergence of Positive Operators

Let X_0, X, and Y satisfy the hypotheses of Theorem 2.1. We wish to investigate the pointwise convergence of positive operators defined on X.

THEOREM 4.1. *Let $\{P_i : i \in I\}$ be a net of positive operators mapping X into Y such that $P_i x$ converges to Px for each x in X_0. Then $P_i x$ converges to Px for each x in X.*

Proof. From the inequality $x_n \leqslant x \leqslant x^n$ we obtain

$$P_i x - Px \leqslant (P_i x^n - Px^n) + (Px^n - Px_n), \qquad \text{and}$$
$$P_i x - Px \geqslant (P_i x_n - Px_n) + (Px_n - Px^n).$$

Since Y has a normal cone, the theorem follows.

As an application of this theorem let X_0 denote the space of all real-valued continuous functions defined on $[0, 1]$, X the space of all real-valued Riemann integrable functions defined on $[0, 1]$ and P the Riemann integral. Define

$$P_n x = \sum_{k=1}^{n} w_{nk} x(t_{nk}), \qquad w_{nk} \geqslant 0, \qquad 0 \leqslant t_{nk} \leqslant 1, \qquad n = 1, 2,\ldots.$$

If $P_n x \to Px$ for all x in X_0, then $P_n x \to Px$ for all x in X. Most of the usual quadrature formulas, Newton–Cotes excepted, have the above properties.

DEFINITION 4.1. Let X be a normed linear space and X' its normed dual. We say that a subset Ω of X' is *norm-determining* if $\|f\| \leqslant 1$ for all f in Ω, and for each x in X

$$\|x\| = \sup\{|f(x)| : f \in \Omega\}.$$

For example the unit ball in X' is a norm-determining subset of X'. If S is a compact metric space, then the point evaluation functionals comprise a norm-determining subset of $C(S)'$. If X is a normed linear space ordered by a normal cone then each continuous linear functional can be written as the difference of two positive linear functionals [9, p. 72], so there exists at least one subset Ω of X' consisting of positive functionals such that $\Omega - \Omega$ is a norm-determining set.

We now prove a result which generalizes Korovkin's theorem [6, p. 14]. For other generalizations see [11, 14].

THEOREM 4.2. *Let X be an ordered Banach space, Y an ordered normed linear space, X_0 a subspace of X and Ω a set of positive continuous linear functionals such that $\Omega - \Omega$ is a norm-determining subset of Y'. Let P be a positive linear operator from X into Y. For each x in X, $n = 1, 2, \ldots$, and t in Ω assume there exist x_{nt} and x^{nt} in X_0 such that*

(i) $x_{nt} \leqslant x \leqslant x^{nt}$,

(ii) $t(Px^{nt} - Px_{nt}) \to 0$ *as* $n \to \infty$, *uniformly for* $t \in \Omega$,

and for each x in X and n

(iii) $\{x_{nt} : t \in \Omega\}$, $\{x^{nt} : t \in \Omega\}$ *are totally bounded.*

Let $\{P_k\}$ be a sequence of positive linear operators on X into Y such that

(iv) $\| P_k x - Px \| \to 0$ as $k \to \infty$, *for each x in X_0.*

Then $\| P_k x - Px \| \to 0$ as $k \to \infty$, for each x in X.

Proof. By the uniform boundedness principle

$$\| P_k x_{nt} - Px_{nt} \| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{uniformly in } t,$$
$$\| P_k x^{nt} - Px^{nt} \| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{uniformly in } t.$$

Then (i), (ii), and (iv) yield

$$t(P_k x - Px) \to 0 \quad \text{as} \quad k \to \infty, \quad \text{uniformly for } t \in \Omega.$$

Since $\Omega - \Omega$ is norm-determining we have $\| P_k x - Px \| \to 0$.

A special case of Theorem 4.2 was obtained by Anselone in [1].

Let R^m denote m-dimensional Euclidean space and $t = (t_1, \ldots, t_m)$ points of R^m. Let F be a compact subset of R^m and $C(F)$ the Banach space of continuous, real-valued functions defined on F with the sup norm.

THEOREM 4.3 (Korovkin). *Let $\{P_k\}$ be a sequence of positive linear operators mapping $C(P)$ into itself. Assume $P_k x \to x$ for the functions $x(t) = 1$, $x(t) = t_i$, $i = 1, 2, \ldots, m$, $x(t) = t_1^2 + \cdots + t_m^2$. Then $P_k x \to x$ for each x in $C(F)$.*

Proof. For each x in $C(F)$, $n = 1, 2,\ldots, t$ in F, there exist x^{nt}, x_{nt} of the form

$$x_{nt}(s) = x(t) - (1/n) - a_n[(s_1 - t_1)^2 + \cdots + (s_m - t_m)^2] \qquad (4.1)$$

$$x^{nt}(s) = x(t) + (1/n) + a_n[(s_1 - t_1)^2 + \cdots + (s_m - t_m)^2] \qquad (4.2)$$

such that

$$x_{nt} \leqslant x \leqslant x^{nt}. \qquad (4.3)$$

This follows from the uniform continuity of x. Note that

$$x^{nt}(t) - x_{nt}(t) = 2/n$$

and that the sets

$$\{x^{nt} : t \in F\}, \qquad \{x_{nt} : t \in F\}$$

are bounded and equicontinuous. Therefore the result follows from Theorem 4.2 with $\Omega = F$ and $P = I$.

5 Weak Convergence and Convergence on $L_p[0, 1]$

We now give some results concerning weak convergence of positive operators. If E is a TVS and x_n, $x \in E$, $n = 1, 2,\ldots$, we shall denote the weak convergence of the sequence $\{x_n\}$ to x by $x_n \to x$ (w).

Let Y be an ordered TVS such that every continuous linear functional on Y can be written as the difference of two positive continuous functionals. The proof of the first result is similar to that of Theorem 4.1.

Proposition 5.1. *Let X be an ordered vector space and $X_0 \subset X$. Let P be a positive operator mapping X into Y such that for $n = 1, 2,\ldots$, there exist x^n, x_n in X_0 with*

(i) $x_n \leqslant x \leqslant x^n$,

(ii) $P(x^n - x_n) \to 0$ (w).

Let $\{P_i : i \in I\}$ be a net of positive operators mapping X into Y such that $P_i x \to Px$ (w) for each x in X_0. Then $P_i x \to Px$ (w) for each x in X.

Theorem 5.1. *Let Ω be a set of positive continuous linear functionals on Y such that for each x in X, $n = 1, 2,\ldots, t$ in Ω, there exist x_{nt}, x^{nt} in X_0 with*

(i) $x_{nt} \leqslant x \leqslant x^{nt}$,

(ii) $t[Px^{nt} - Px_{nt}] \to 0$ *as* $n \to \infty$.

Let $\{P_i : i \in I\}$ *be a net of positive operators from* X *into* Y *such that*

(iii) $t[P_i x - Px] \to 0$ *for each* x *in* X_0, t *in* Ω.

Then $t[P_i x - Px] \to 0$ *for each* x *in* X, t *in* Ω.

Proof. This follows from the inequalities

$$t[P_i x - Px] \leqslant t[P_i x^{nt} - Px^{nt}] + t[Px^{nt} - Px_{nt}],$$
$$t[P_i x - Px] \geqslant t[P_i x_{nt} - Px_{nt}] + t[Px_{nt} - Px^{nt}],$$
$$t \in \Omega, \qquad n = 1, 2, \ldots .$$

Suppose F is a compact subset of R^m and $\{P_k\}$ is a sequence of positive linear operators mapping $C(F)$ into itself.

COROLLARY 5.1. *If the norms* $\| P_k \|$, $k = 1, 2, \ldots$, *are uniformly bounded and* $(P_k x)(t) \to x(t)$, $t \in F$, *for the functions* $x(t) = 1, t_1, t_2, \ldots, t_m, t_1^2 + \cdots + t_m^2$, *then* $(P_k x)(t) \to x(t)$, $t \in F$, *for all* x *in* $C(F)$.

Proof. This follows from Eqs. (4.1), (4.2), Theorem 5.1 and the fact that for bounded sequences in $C(F)$, weak convergence is equivalent to pointwise convergence.

COROLLARY 5.2. *Let* $C(R^m)$ *denote the vector space of all real-valued continuous functions defined on* R^m *and* $\{P_k\}$ *a sequence of positive linear operators mapping* $C(R^m)$ *into itself. Suppose for each compact subset* F *of* R^m, *there exists a constant* $M(F)$ *such that* $|P_k(1)(t)| \leqslant M(F)$, *for* t *in* F, $k = 1, 2, \ldots$. *If* $(P_k x)(t) \to x(t)$, $t \in R^m$, *for the functions* $x(t) = 1, t_1, \ldots, t_m$, $t_1^2 + \cdots + t_m^2$, *then* $(P_k x)(t) \to x(t)$ *for each* x *in* $C(R^m)$.

Now we shall consider the convergence of positive operators on $L_p[0, 1]$, $1 \leqslant p < \infty$.

THEOREM 5.2. *Let* $\{P_k\}$ *be a sequence of positive linear operators mapping* $L_p[0, 1]$ *into itself. Suppose* $P_k x \to x\ (w)$ *for the functions* $x(t) = 1, t, t^2$. *Then* $P_k x \to x\ (w)$ *for every* x *in* $L_p[0, 1]$ *if and only if the norms* $\| P_k \|$, $k = 1, 2, \ldots$, *are uniformly bounded.*

Proof. If $P_k x \to x\ (w)$ for each x in $L_p[0, 1]$, then the norms would be uniformly bounded by the uniform boundedness principle. Conversely suppose there is a constant M such that $\| P_k \| \leqslant M$, $k = 1, 2, \ldots$. It suffices to prove the theorem for continuous functions so let $x \in L_p[0, 1]$, x continuous and define $\varphi_t(s) = (t - s)^2$, $0 \leqslant s, t \leqslant 1$. For each $n = 1, 2, \ldots$, and each t in $[0, 1]$ we have x^{nt}, x_{nt} as defined in Eqs. (4.1), (4.2) (with $m = 1$) and satisfying (4.3). Let $f \in L_p[0, 1]' = L_q[0, 1]$, $(1/p) + (1/q) = 1$ and we may assume $f(t) \geqslant 0$ a.e. For each integer k and n we have

$$\int_0^1 f(t)[(P_k x)(t) - x(t)]\,dt \leqslant \int_0^1 x(t) f(t)[(P_k 1)(t) - 1]\,dt + \frac{1}{n}\int_0^1 f(t)[(P_k 1)(t)]\,dt + a_n \int_0^1 f(t)[(P_k \varphi_t)(t)]\,dt,$$

$$\int_0^1 f(t)[(P_k x)(t) - x(t)]\,dt \geqslant \int_0^1 x(t) f(t)[(P_k 1)(t) - 1]\,dt - \frac{1}{n}\int_0^1 f(t)[(P_k 1)(t)]\,dt - a_n \int_0^1 f(t)[(P_k \varphi_t)(t)]\,dt,$$

$$\int_0^1 f(t)[(P_k \varphi_t)(t)]\,dt \to 0.$$

Hence $f(P_k x) \to f(x)$ and the proof is complete.

Now we are in a position to give a Korovkin-type theorem for $L_p[0, 1]$.

THEOREM 5.3. *Let $\{P_k\}$ be a sequence of positive linear operators mapping $L_p[0, 1]$ into itself. If*

(i) *the norms $\| P_k \|$, $k = 1, 2, \ldots$, are uniformly bounded,*

(ii) $P_k 1 \to 1$,

(iii) *$P_k x \to x$ (w) for the functions $x(t) = t$, t^2, then $P_k x \to x$ for all x in $L_p[0, 1]$.*

Proof. Let G be the set of all g in $L_p[0, 1]$ such that g is the characteristic function of a subinterval of $[0, 1]$ or the characteristic function of the complement of such a subinterval. It suffices to show $P_k g \to g$ for each g in G. For g in G, let $Z_g = \{t : g(t) = 0\}$. By Theorem 5.3 we have $P_k g \to g$ (w) which implies

$$\int_{Z_g} (P_k g)(t)\,dt \to 0 \quad \text{as} \quad k \to \infty, \qquad g \in G. \tag{5.1}$$

Let g in G be fixed and $f = 1 - g$. Then $f \in G$ and

$$\int_{Z_f} (P_k f)(t)\,dt = \int_{Z_f} [(P_k 1)(t) - 1]\,dt + \int_{Z_f} [1 - (P_k g)(t)]\,dt.$$

Then (5.1) yields

$$\int_{Z_f} | 1 - (P_k g)(t)| \, dt \to 0 \qquad \text{as} \quad k \to \infty.$$

Another application of (5.1) gives

$$\int_0^1 |(P_k g)(t) - g(t)| \, dt \to 0 \qquad \text{as} \quad k \to \infty.$$

Let $\{P_{k'} g\}$ be an arbitrary subsequence of $\{P_k g\}$. Then there is a further subsequence $\{P_{k''} g\}$ such that $(P_{k''} g)(t) \to g(t)$ a.e., as $k'' \to \infty$. Let $\epsilon > 0$ be given. Then by Egoroff's theorem there exists a set A with $m(A) < \frac{1}{4}\epsilon[2^p + 2^{2p}]^{-1}$ and on $[0, 1] - A$, $(P_{k''} g)(t) \to g(t)$ uniformly. The inequality

$$\int_A |(P_{k''} g)(t) - g(t)|^p \, dt \leqslant m(A)[2^p + 2^{2p}] + 2^{2p}(\| P_k 1 - 1 \|)^p$$

yields $\| P_{k''} g - g \| < \epsilon$ for k'' sufficiently large. Therefore $P_k g \to g$ and the proof is finished.

A similar result was established in [4] using the three functions $x(t) = 1$, $\sin t$, $\cos t$, except it was assumed that $P_k x$ converges strongly to x for all three functions.

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